4.1 (I) SQUARE ROOTS AND CUBE ROOTS

Activity – Do the Investigation on page 152 #1-2. Then answer the questions below.

3. Reflect and Respond. Discuss with a partner and write your answer here.

a) What strategy could you use to find the side length of a square if you were given the area?

b) What strategy could you use to find the edge length of a cube if you were given the volume?

c) Explain, using a diagram, how you could predict:

- the side length of a square with an area of 64 square units

- the edge length of a cube with a volume of 343 cubic units

Squares and Cubes

In this chapter we will be exploring exponents and roots. To begin, we will start with second and third powers and roots.

Write down a definition for each term in your own words. Include examples.

Perfect Square:
Square Root:

Perfect Cube:

Cube Root:

Symbols and Terminology

The entire expression is called a _________________ ______.

The symbol, \( \sqrt[3]{ } \), is called the _______________.

The expression inside the sign is called the _________________.

The little number on the outside of the radical sign is called the ________________.

What does this number tell you?

What does it mean if there is nothing there?

Prime Factorization

For some smaller numbers it is fairly easy to tell if they are a perfect square or a perfect cube. For larger numbers it is easier to use methods other than memorization or guess and check. One method is called PRIME FACTORIZATION. This method involves writing a number as a product of its prime factors.

What is a prime number?

Give some examples:
What is a factor?

**Ex.** What is the prime factorization of 225?

\[
225 = 3 \times 3 \times 5 \times 5
\]

We can write the factors of 225 in two equal groups, therefore 225 is a perfect square.

**Ex.** What is the prime factorization of 72?

\[
72 = 3 \times 3 \times 2 \times 2 \times 2
\]

Is 72 a perfect square, perfect cube, both or neither?
Ex. What is the prime factorization of 4096? Is it a perfect square, a perfect cube, both or neither?
4.1 (II) SQUARE ROOTS AND CUBE ROOTS – Problems (and Review)

Activity: With a partner, do question #20 on page 161. The question is copied below.

<table>
<thead>
<tr>
<th>Number</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number Squared</td>
<td>0</td>
<td>1</td>
<td>16</td>
<td>25</td>
<td>64</td>
<td></td>
</tr>
</tbody>
</table>

b) Based on the table, how would you label the axes on the graph?

c) What does each small unit represent on the horizontal axis? On the vertical axis?

d) Explain how you could use the graph to find the value for $5^2$. 
e) How could you use the graph to evaluate $\sqrt{49}$?

f) Show how you could use the graph to determine the approximate value for $\sqrt{18}$. Multiply your answer by itself. How close is your product to 18?

g) What is an approximation for $(5.2)^3$?

We’ll do some more examples with square roots and cube roots.

**Ex.** Estimate the square root of 165.

- What two perfect squares does this number lie between?
- Which perfect square is it closest to?
- What is the approximate square root of 165?
- What is the square root of 165 to the nearest hundredth?

**Ex.** Estimate the cube root of 89.

- What two perfect cubes does this number lie between?
- Which perfect cube is it closest to?
- What is the approximate cube root of 89?
- What is the cube root of 89 to the nearest hundredth?
### EXponent Laws

*Note that $a$ and $b$ are rational or variable bases and $m$ and $n$ are integral exponents.

<table>
<thead>
<tr>
<th>Law</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Product of Powers</td>
<td>$a^m \cdot a^n = a^{m+n}$</td>
</tr>
<tr>
<td>Quotient of Powers</td>
<td>$\frac{a^m}{a^n} = a^{m-n}$, $a \neq 0$</td>
</tr>
<tr>
<td>Power of a Power</td>
<td>$(a^m)^n = a^{mn}$</td>
</tr>
<tr>
<td>Power of a Product</td>
<td>$(ab)^n = a^n b^n$</td>
</tr>
<tr>
<td>Power of a Quotient</td>
<td>$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, $b \neq 0$</td>
</tr>
<tr>
<td>Zero Exponent</td>
<td>$a^0 = 1$, $a \neq 0$</td>
</tr>
</tbody>
</table>

**Simplify.**

a) $(-3xy^5)^2$

b) $\left(\frac{2xy^3}{6x^4y}\right)^3$

c) $(-2m^2n^3)^2(3mn^5)$

d) $\left(\frac{3y^3}{2x^4}\right)^3$
Activity: With a partner do the activity below.

1) On the side margin of this paper, draw a line 16 cm long and mark it as shown on page 163.

2) Mark a point halfway between 0 and 16. Label the point with its value and its equivalent value in exponential form \(2^x\). Repeat this procedure until you reach a value of 1 cm.

   a. How many times did you halve the line segment to reach 1 cm?

   b. What do you notice about the exponents as you keep reducing the line segment by half?

3) 

   a. Mark the halfway point between 0 and 1. What fraction does this represent?

   b. Using the pattern established in step 2, what is the exponential form of the fraction?

   c. Halve the remaining line segment two more times.

4) Use a table to summarize the line segment lengths and the equivalent exponential form in base 2.

<table>
<thead>
<tr>
<th>Segment Length</th>
<th>Exponential Form</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
5) Reflect and Respond.

a. Describe the pattern you observe in the exponents as the distance in halved.

b. Is there a way to rewrite each fraction so that it is expressed as a power with a positive exponent? Try it. Compare this form to the equivalent power with a negative exponent. What is the pattern?

c. Create a general form for writing any power with a negative exponent as an equivalent power with a positive exponent.

6)

a. Carbon-14 has a half-life of 5700 years. This means the rate of decay is \( \frac{1}{2} \) or \( 2^{-1} \) every 5700 years. What fraction of carbon-14 would be present in organic material that is 11,400 years old? 17,100 years old? Express each answer as a power with a negative exponent. Explain how you arrived at your answers.

b. Suggest types of situations when a negative exponent might be used.
All of the exponent laws are useful for simplifying expressions with integral exponents. It is also useful to be able to change an expression with a negative exponent to an expression with a positive exponent.

\[ a^{-n} = \frac{1}{a^n}, \quad a \neq 0 \]

Example:

\[ \frac{1}{a^{-n}} = a^n, \quad a \neq 0 \]

Example:

For example, there are two methods we could use to simplify the expression \((x^7x^{-4})\)

Method 1: add exponents

\[(x^7x^{-4}) = x^{7+(-4)} = x^4\]

Method 2: use positive exponents

\[(x^7x^{-4}) = (x^7\left(\frac{1}{x^4}\right)) = x^7 \div x^4 = x^{7-4} = x^4\]

Practice. Simplify each expression. State all answers using positive exponents.

a) \((a^4)(a^{-9})\)

b) \(\left(\frac{a^{-3}}{a^2}\right)^4\)
c) \( \left( \frac{6}{5} \right)^{-2} \) 

\( \left( \frac{1}{2} \right)^{-3} \)

Simplify, then evaluate. Express your answers to four decimal places when necessary.

\( (0.2^3)^{-1} \) 

\( \left( \frac{5}{3^3} \right)^{-5} \)
4.2 (II) APPLYING INTEGRAL EXPONENTS

What is the difference between “simplify” and “evaluate”?

Ex. A farmer counted 300 billion blades of grass in his field. If the field is 2500 m², how many blades of grass are there per square meter?

Method 1: Algebra  
Method 2: Exponent Rules

\[
\frac{300,000,000,000}{2500} = \frac{(300)(10^9)}{(25)(10^3)} = \]

Ex. The bacteria Mathematria has a population growth modelled by the formula \( P = 178(1.052)^n \) where \( P \) is the population and \( n \) is the number of hours.

a) How many bacteria were present when the scientist started the experiment (ie. When zero hours had passed?)

b) How many bacteria will there be after 12 hours have passed?
Ex. The half-life of a radioactive material known as Calc II is 23 days. A radiologist starts with 50 mg of the material. The formula that represents this half life is \( A = 50 \left( \frac{1}{2} \right)^{\frac{d}{23}} \), where \( A \) is the amount of material remaining and \( d \) is the number of days elapsed.

a) How much material is left after 92 days?

b) After how many days is only 12.5 mg left?
4.3 (I) RATIONAL EXPONENTS

Investigation: Do the following with a partner. Make sure you each record your responses.

1. According to the product rule for powers

\[(9^\frac{1}{2})(9^\frac{1}{2}) = 9^{\frac{1}{2} + \frac{1}{2}}\]

Then:

\[9^1 = 9^{\frac{1}{2} + \frac{1}{2}}\]

\[= 9^1\]

\[= 9\]

So what is the value of \(9^{\frac{1}{2}}\)?

Use your calculator to check (ask if you’re not sure how).

2. Predict values for

   a) \(4^{\frac{1}{2}}\)
   b) \(16^{\frac{1}{2}}\)
   c) \(36^{\frac{1}{2}}\)
   d) \(49^{\frac{1}{2}}\)

Use your calculator to check your answer.

3. Predict the value of \(8^{\frac{1}{3}}\).

Explain your thinking.

Check your prediction.

4. Reflect and Respond

   a. Explain how determining \(49^{\frac{1}{2}}\) and your definition for square root are related.
b. Express the 12th root of 2 as a power. Evaluate using your calculator and express the answer to six decimal places.

c. Use your calculator to determine the 12th power of this decimal value. Why isn’t the answer 2?

5. Based on what you have just explored, evaluate \(32^{\frac{1}{3}}\) and explain in your own words how you found it.

Further thought: if \(8^{\frac{1}{3}} = \left(\frac{1}{8}\right)^{\frac{1}{3}} \cdot \left(\frac{1}{8}\right)^{\frac{1}{3}} \cdot \left(\frac{1}{8}\right)^{\frac{1}{3}}\) and \(8^{\frac{1}{3}} = 2\), what is \(8^{\frac{2}{3}}\)? Explain why and how you did this.

The same exponent laws that apply to integral exponents also apply to rational exponents.

Ex. Simplify (write each as a power with a single exponent in simplified form).

a) \((a^{\frac{2}{3}})(a^{\frac{2}{3}})\)   c) \((a^{-\frac{2}{3}})(a^{\frac{2}{3}})\)
Ex. Write each expression as a power with a single, positive exponent. Then evaluate (where possible).

b) \( \frac{5^{-0.75}}{5^4} \)

d) \( \frac{8^{1.8}}{16^{0.3}} \)

---

Ex. Write each expression as a power with a single, positive exponent. Then evaluate (where possible).

a) \( \frac{x^{\frac{9}{10}}}{x^\frac{1}{5}} \)

b) \( [(2^3)(3^\frac{3}{4})]^6 \)

c) \( (27x^9)^\frac{2}{3} \)

d) \( \left( \frac{125}{y^5} \right)^{\frac{4}{3}} \)

e) \( \frac{(8^3)(32^\frac{5}{3})}{4^5} \)

f) \( \frac{3^x}{5^y} \)
4.3 (II) RATIONAL EXPONENTS

Assignment: Do the following questions. Show your work!

a) \( (3a^4b^3)(-4ab^2) \)

b) \( \left( \frac{3^\frac{3}{2} \cdot 1^\frac{1}{3}}{r^2 \cdot t^3} \right)^6 \)

c) \( (4m^6n^2)^\frac{1}{2} (3mn^3) \)

d) \( \left( \frac{3y^{0.3}}{2x^{1.4}} \right)^5 \)

e) \( \left( \frac{-8a^3b}{a^2b^2} \right)^\frac{2}{3} \)

f) \( 3(pq)^0 \)

g) \( xy^{-2} \)

h) \( \left( \frac{3a^2}{2b^{-1}} \right)^{-7} \)

i) \( \frac{2^{5n+3}}{2^{n-1}} \)

j) \( (a^x)^x \)
Explain how you did (h) and (i) in your own words.

**Ex.** A cube has a volume of 343 cm\(^3\).

a) Write a power which represents the edge length of the cube.

b) Write a power which represents the surface area of the cube.

c) Calculate the exact edge length and surface area of the cube.

**Ex.** Grant invests $2000 in an account that has an interest rate of 8.4% per year. The bank updates the account quarterly using the formula 

\[ A = P \left(1 + \frac{r}{100}\right)^q \]

where \( q \) represents the number of quarterly periods and \( A \) represents the amount in the account.

a) What is the relationship between the 8.4% and the 1.084 in the formula?

b) What is the value of the investment after the 3\(^{rd}\) quarter?

c) What is the value of the investment after 3 years?
**Ex.** The number of bacteria in a culture grows according to the formula \( P = 10000(3)^{\frac{h}{10}} \) when the initial number of bacteria is 10000 and where \( P \) is the number of bacteria and \( h \) is the number of hours that have passed.

a) What does 10000 represent in the formula?

b) How many bacteria will be present in 15 hours?

c) How many bacteria will be present in 30 hours?

d) What can you say about the rate of growth of the bacteria? How is this expressed in the formula?

e) About how long will it take for there to be more than a million bacteria?
4.4 (I) IRRATIONAL NUMBERS

Numbers are classified into sets according to common characteristics. All numbers we talk about will fall under the category of real numbers.

**Real Numbers,** \( R = \{\text{all irrational and rational numbers}\} \)

<table>
<thead>
<tr>
<th><strong>Rational Numbers,</strong> ( Q )</th>
<th>( {\text{all numbers that can be written as a fraction in the form } \frac{a}{b} \text{ where } a \text{ and } b \text{ are integers and } b \neq 0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Integers</strong> ( I )</td>
<td>( {\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots} )</td>
</tr>
<tr>
<td>The set of integers also includes all the whole numbers.</td>
<td></td>
</tr>
<tr>
<td><strong>Whole Numbers</strong> ( W )</td>
<td>( {0, 1, 2, 3, \ldots} )</td>
</tr>
<tr>
<td>The set of whole numbers also includes all the natural numbers.</td>
<td></td>
</tr>
<tr>
<td><strong>Natural Numbers</strong> ( N )</td>
<td>( {1, 2, 3, \ldots} )</td>
</tr>
<tr>
<td>These numbers are also called counting numbers.</td>
<td></td>
</tr>
</tbody>
</table>

Numbers can be irrational, or rational, but not both.
Powers with exponents that are rational (can be written as fractions) can also be written in radical form.

Put the numbers at the right into the smallest possible correct circle.

**Terminology**

We discussed some terminology earlier in this unit. Using your textbook or some other resource, write down a definition for each term in your own words and label the expression below.

\[
\frac{10}{7}, \quad -\sqrt{5}, \\
0, \quad 2\sqrt{2}, \\
-3.45454545..., \quad \pi, \quad -\frac{2}{9}, \\
-1, \quad 3
\]

\[\sqrt[3]{7}
\]

**Radical:**

**Radicand:**

**Index:**

Powers with exponents that are rational (can be written as fractions) can also be written in radical form.
For example:

\[ a^{\frac{3}{2}} = \sqrt{a^3} \quad \text{or} \quad a^{\frac{1}{2}} = (\sqrt{a})^2 \]

Label the numerator and denominator of the exponent as either the power or the root. Where does the power go when the expression is in radical form? Where does the root go? How can you remember this?

Using this example, can you write the following in radical form?

a) \( 6^{\frac{1}{3}} \)  
b) \( -5^{\frac{1}{4}} \)  
c) \( (-3)^{\frac{1}{5}} \)  
d) \( 1^{\frac{1}{11}} \)

Write the following in exponential form.

a) \( \sqrt[7]{7} \)  
b) \( \sqrt[3]{-3} \)  
c) \( \sqrt[4]{16x^3} \)  
d) \( (\sqrt[5]{5})^4 \)

Evaluate. Round to four decimal places if necessary.

a) \( 1.44^{\frac{1}{2}} \)  
b) \( \sqrt[3]{-125} \)  
c) \( \frac{1}{\sqrt[4]{16}} \)  
d) \( \frac{0.2\sqrt{7}}{\sqrt{3}} \)
4.4 (II) IRRATIONAL NUMBERS

Consider the fraction $\frac{38}{12}$. Can this be simplified? In what other forms can it be written?

We can do the same with radicals. Radicals can be written as **entire radicals** or **mixed radicals**.

Write down your own definition for each term and give an example:

**Entire radical:**

**Mixed radical:**

Evaluate to four decimal places:  

a) $\sqrt{54}$  

b) $3\sqrt{6}$

What do you notice? Why do you think this is?

Can you write $2\sqrt{5}$ as an entire radical? How?

Use your calculator to check that both expressions have the same value.

Write the following as entire radicals. Then arrange from least to greatest.

a) $2\sqrt{5}$  

b) $3\sqrt{8}$  

c) $9\sqrt{2}$  

d) $4\sqrt{3}$
Can you write \(3\sqrt[7]{7}\) as an entire radical? How?

Use your calculator to check that both expressions have the same value.

Write the following as entire radicals.

a) \(2\sqrt[3]{11}\)  
b) \(3\sqrt[3]{5}\)  
c) \(9\sqrt[3]{2}\)  
d) \(8\sqrt[3]{3}\)

**Ex.** Using what you have learned, can you write \(\sqrt{20}\) as a mixed radical? How?

Use your calculator to check that both expressions have the same value.

**Ex.** Using what you have learned, can you write \(\sqrt[3]{48}\) as a mixed radical? How?

Use your calculator to check that both expressions have the same value.

Write the following as mixed radicals. It may be useful to make a factor tree.

a) \(\sqrt{72}\)  
b) \(\sqrt{54}\)  
c) \(\sqrt[4]{162}\)  
d) \(\sqrt{63}\)
4.4 (III) IRRATIONAL NUMBERS - Ordering

Ordering Radicals

To be able to order radicals we need to have them in comparable forms.

For example, which is bigger: $2.\sqrt{3}$ or $\sqrt{5}$?

Ex. Write each number as an entire radical in order to determine the order from least to greatest.

$7\sqrt{6}, 6\sqrt{7}, 12\sqrt{3}, 11\sqrt{5}$

Ex. When a satellite is $h$ kilometres above the Earth, the time, $t$, in minutes, to complete one orbit is given by the formula

$$t = \frac{\sqrt{(6370 + h)^2}}{6024}$$

a) A telecommunications satellite is placed 30 km above the Earth. How long does it take the satellite to make one orbit?

b) A satellite is placed in geosynchronous orbit about Earth (this means it takes the same amount of time to complete one orbit as the Earth does to complete one rotation). What must its altitude be?
**ADDING AND SUBTRACTING RADICALS**

Adding/subtracting monomial terms: only like terms can be combined.

\[ 7x + 5y + 9x \text{ simplifies to } 16x + 5y \]

Adding/subtracting radical terms: only terms with the same radical can be combined.

\[ 7\sqrt{2} + 5\sqrt{3} + 9\sqrt{2} \text{ simplifies to } 16\sqrt{2} + 5\sqrt{3} \]

**exercise:** Simplify \( \sqrt{11} - 8\sqrt{13} - 5\sqrt{11} - 8\sqrt{13} \)  \[ \text{Answer: } -4\sqrt{11} - 16\sqrt{13} \]

**exercise:** Simplify \( \sqrt{5} + \sqrt{20} - \sqrt{45} \)  \[ \text{Answer: } 0 \]

**exercise:** Simplify \( 8\sqrt{6} - 5\sqrt{12} + 2\sqrt{27} \)  \[ \text{Answer: } 8\sqrt{6} - 4\sqrt{3} \]
Adding polynomials: only like terms can be combined.

\[(2x + 3y) + (5x - 4y)\] is rewritten as \[2x + 3y + 5x - 4y\]

Similarly

\[(2\sqrt{2} + 3\sqrt{10}) + (5\sqrt{2} - 4\sqrt{10})\] becomes \[2\sqrt{2} + 3\sqrt{10} + 5\sqrt{2} - 4\sqrt{10}\]

exercise: Simplify \((3\sqrt{48} - 4\sqrt{8}) + (4\sqrt{27} - 2\sqrt{72})\) \[Answer: 24\sqrt{3} - 20\sqrt{2}\]

Subtracting polynomials: add the opposite.

\[(2x + 3y) - (5x - 4y)\] is rewritten as \[(2x + 3y) + (-5x + 4y)\]

Similarly

\[(2\sqrt{2} + 3\sqrt{10}) - (5\sqrt{2} - 4\sqrt{10})\] becomes

\[(2\sqrt{2} + 3\sqrt{10}) + (-5\sqrt{2} + 4\sqrt{10})\]

exercise: Simplify \((3\sqrt{48} - 4\sqrt{8}) - (4\sqrt{27} - 2\sqrt{72})\) \[Answer: 4\sqrt{2}\]
MULTIPLYING RADICALS

The rule for multiplying radicals is \( \sqrt[n]{ab} = \left( \sqrt[n]{a} \right) \left( \sqrt[n]{b} \right) \)

The first condition for simplest radical form with square roots is:

- the radicand cannot have a factor that is perfect square larger than 1.

examples: Simplify

a) \( (4\sqrt{3})(11\sqrt{2}) \)
\[
4 \times \sqrt{3} \times 11 \times \sqrt{2} = 44 \times \sqrt{3 \times 2} = 44\sqrt{6}
\]

b) \( (2\sqrt{33})(5\sqrt{77}) \)
\[
2 \times \sqrt{33} \times 5 \times \sqrt{77} = 10 \times \sqrt{3 \times 11 \times 7 \times 11} = 10 \times 11 \times \sqrt{3 \times 7} = 110\sqrt{21}
\]

e exercises: Simplify

a) \( (\sqrt{10})(\sqrt{13}) \) \( \sqrt{130} \)
b) \( (2\sqrt{7})(3\sqrt{7}) \) \( 42 \)
c) \( (\sqrt{30})(\sqrt{42}) \) \( 6\sqrt{35} \)
d) \( (8\sqrt{3})(2\sqrt{6}) \) \( 48\sqrt{2} \)
DIVIDING RADICALS

The rule for dividing radicals is \( \sqrt[n]{\frac{a}{b}} = \frac{n\sqrt{a}}{n\sqrt{b}} \)

The three conditions for simplest radical form are:
1. the radicand cannot have a factor that is perfect square larger than 1.
2. the radicand cannot be a fraction or decimal,
3. the denominator cannot contain a radical.

examples: Simplify

a) \( \frac{\sqrt{70}}{\sqrt{35}} = \frac{\sqrt{70}}{\sqrt{35}} \)

   = \( \sqrt{2} \)

b) \( \sqrt{\frac{10}{18}} = \)